A single-spin precessing gravitational wave in closed form

Andrew Lundgren[1] and R. O'Shaughnessy[2]

[1] *Albert-Einstein-Institut, Callinstr. 38, 30167 Hannover, Germany*

[2] *Center for Gravitation and Cosmology, University of Wisconsin-Milwaukee, Milwaukee, WI 53211, USA*

Current searches for gravitational waves from compact binaries use templates where the binary does not precess. By contrast, generic black hole-neutron star binaries will significantly precess, inducing modulations that nonprecessing searches are particularly ill-suited to recover. In this paper we provide a closed-form representation of the single-spin precessing waveform in the frequency domain by reorganizing the signal as a sum over harmonics, each of which resembles a nonprecessing waveform. This form enables simple analytic calculations (e.g., a Fisher matrix) and computationally fast frequency-domain templates. We have verified that our `SpinTaylorF2SingleSpin` model is faithful to the time-domain `SpinTaylorT2` model: for the majority of generic BH-NS binaries, the corresponding signals generated from each model agree to better than 1%. Our waveform model is now available as part of the publicly-available `lalsimulation` waveform-generation code.

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For the astrophysically most plausible strong gravitational wave sources – coalescing compact binaries of black holes and neutron stars with total mass $M = m_1 + m_2 \leq 16 M_\odot$ – ground based gravitational wave detector networks (notably LIGO [1] and Virgo [2, 3]) are principally sensitive to each binary's nearly-adiabatic and quasi-circular inspiral [4–12]. For nonprecessing binaries, the inspiral signal model has been explored in detail, both analytically and with numerical simulation. Fast, stationary-phase approximations to the signal exist which faithfully reproduce the signal [13]. By contrast, plausible astrophysical processes will produce merging black hole binaries with arbitrary spin orientations, fully populating the 15-dimensional model space for quasi-circular inspiral. Fully generic binaries undergo extremely complicated evolution, consisting of a gradually decaying and rapidly precessing orbit [14–16]. For many astrophysically interesting sources, including many binaries containing at least one black hole, significant spin precession is expected while the signal passes through LIGO and Virgo's sensitive frequency band. Precessing sources have been substantially less-thoroughly modeled. At present, the current state-of-the-art `SpinTaylorT4` [5, 6] and `SpinTaylorT2` models [13] solve a coupled time-domain ordinary differential equation to evolve the orbits and spins. Until this work, no fast, stationary-phase approximation existed which faithfully reproduces a precessing gravitational waveform. The need to numerically evolve several differential equations imposes a significant computational cost. The added physical parameters needed to capture precessing spins multiply the computational burden of using precessing signals for data analysis. On the one hand, search strategies attempting to recover precessing signals could not employ the full signal, instead adopting a *detection template family*, whose waveforms usually at best imperfectly approximate the target waveforms [5, 7, 8, 10, 17] but whose template banks are smaller and better understood. On the other hand, the large signal space required an equally large number of slow-to-evaluate templates to cover it, placing a similarly-large computational burden on gravitational wave parameter estimation using physical signals. Finally, the Fisher matrix and other quantities arising naturally in data analysis take a simpler form in our model, where derivatives can be efficiently evaluated [18]. By contrast, time-domain models are essentially opaque to analysis, since interesting calculations can only be done numerically.

In this paper, we show that the gravitational wave signal for a single-spin binary can be well approximated with a simple analytic form in the frequency domain. Our decomposition expresses the signal as a sum of five terms, each a weak modulation (sideband) of a leading-order ("carrier") function:

$$\widetilde{h}_+(f) = A(f) \sum_{\mathrm{m}=-2}^{2} E_{\mathrm{m}}(f)\, e^{i(\mathrm{m}\alpha(f)+2\Psi(f))} \,. \qquad (1)$$

Critically, our decomposition provides closed-form functions of frequency for all of the terms entering this expression. $E_{\mathrm{m}}(f)$ can be thought of as a complex antenna pattern which contains all of the orientation dependence of the waveform, and evolves very slowly over radiation reaction rather than orbital timescales.

We have tested our frequency-domain `SpinTaylorF2SingleSpin` signal model to the time-domain `SpinTaylorT2` approximant [13] by calculating the faithfulness. The noise-weighted inner product of two waveforms h_1 and h_2 is [19]

$$(h_1|h_2) = 4\mathcal{R}\int_{f_l}^{f_h} \frac{\tilde{h}_1(f)\tilde{h}_2^*(f)}{S_n(f)} \,, \qquad (2)$$

and the overlap is $(h_1|h_2)/\sqrt{(h_1|h_1)(h_2|h_2)}$. We generate both signals with the exact same mass, spin, and orientation parameters, then maximize the overlap over only the time and phase, yielding the so-called faithfulness. The same polarization of the two waveforms is compared. This is a more stringent requirement than requiring that our model be effectual (high overlap when

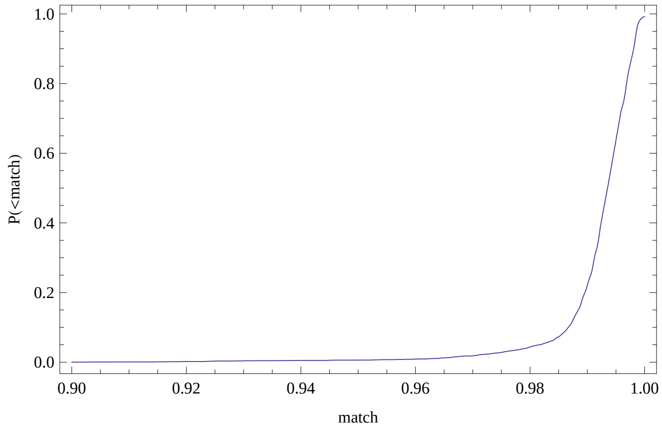

FIG. 1: **Faithfulness of our model**: For $10+1.4M_\odot$ BH-NS binaries with random orientations and random BH spins with magnitudes in [0.5,1], we compute the faithfulness (overlap maximized over time and phase) of our `SpinTaylorF2SingleSpin` model with `SpinTaylorT2`. The agreement is better that 2% except for a small number of cases with very large spin-orbit misalignment.

maximizing over parameters like masses, as in a search). We did 5000 simulations of $10+1.4M_\odot$ BH-NS binaries with random source orientations, and BH spin with random direction and magnitudes $\chi \in [0.5,1]$. For $S_n(f)$, we use the Advanced LIGO high-power zero-detune noise spectrum[20, 21] with $f_l = 15Hz$ and the ISCO frequency as f_h. The two models agree to better than 2% in almost all cases, with mismatches of up to 10% occuring only for strong spin-orbit misalignment ($\hat{L}\cdot\hat{S} = \kappa < -0.9$). This demonstrates that the approximations that went into building our model are all justified.

In our notation, the components of the binary have masses m_1 and m_2, with m_1 the larger mass; for convenience, we also use $M = m_1+m_2$, $\eta = m_1 m_2/M^2$, and the mass ratio $q = m_2/m_1$. Only the larger mass has spin, with a dimensionless spin parameter χ_1 and spin $|\mathbf{S}_1| = m_1^2\chi_1$. The orbital angular velocity is denoted ω and is related to the gravitational wave frequency by $\omega = \pi f$ and to the orbital velocity by $v = (M\omega)^{1/3}$.

Because the waveform only has two independent polarizations along any line of sight, we can without loss of generality treat it as overhead of the detector. Any other sky position just acts as an amplitude scaling and a rotation of the polarization. To define the orientation of the binary with respect to the detector, we use the total angular momentum $\mathbf{J} = \mathbf{L}+\mathbf{S}$ which is nearly fixed during the inspiral. The inclination of $\mathbf{J}$ to the line toward the detector is θ, and the polarization angle ψ is defined by the angle between $\mathbf{J}$ and the x-arm of the detector in the plane of the sky; see [22], [23], and references therein.

The precessing waveform can be decomposed into a non-precessing waveform acted on by a time-dependent rotation [24–27]. The rotation accounts for the precession of the orbital angular momentum of the binary. The waveform in the source frame is

$$\widetilde{h}_+ - i\widetilde{h}_\times = \sum_{\ell,m'} \tilde{h}^{\ell,m'}(t)\,{}_{-2}\tilde{Y}_{\ell,m'}(\widetilde{\theta},\widetilde{\phi})e^{im'\Phi} \ . \tag{3}$$

where Φ is the orbital phase and where here and henceforth $\tilde{\cdot}$ denotes the frame co-rotating with the precession of $\mathbf{L}$. In this frame, the source amplitudes nearly obey $\widetilde{h}_{\ell,-m} = (-1)^\ell \widetilde{h}^*_{\ell,m}$, to the extent spin-dependent higher harmonics can be neglected; expressions for $\tilde{h}^{lm}$ exist in the literature [4].

The source frame and inertial frame are related by a time-dependent rotation, specified by three Euler angles (α,β,γ).[1] The source frame is aligned with the instantaneous angular momentum $\mathbf{L}$; the inertial frame is aligned with the total angular momentum $\mathbf{J}$ of the binary. In addition to the two Euler angles (α,β) needed to map $\mathbf{J}\to\mathbf{L}$, we apply a third Euler angle $\gamma \equiv -\int \cos\beta(d\alpha/dt)dt$ to minimize superfluous coordinate changes associated with the orbital plane; see Boyle et al. [24]. By applying this time-dependent rotation, the waveform in an *inertial* frame can be expressed as a weighted sum of the co-rotating-frame amplitudes $\tilde{h}^{lm}_S$, this rotation allows can be decomposed as the waveform in a frame co-rotating with the precession and a time-dependent transformation from the co-rotating to the detector coordinates.

$$h_+ - ih_\times = \sum_{\ell,m',m} D^{(\ell)}_{m',m}(\alpha,\beta,\gamma)\tilde{h}^{\ell,m}(t)\,{}_{-2}Y_{\ell,m'}(\theta,\phi)e^{im\Phi} \tag{4}$$

where $D^{(\ell)}_{m'm}$ is the usual Wigner rotation matrix representation of SU(2) [24–26]. Because $h_+ - ih_\times$ has spin weight -2, this expression must be proportional to $\exp(-2i\psi)$, where the polarization angle ψ is the angle of $\mathbf{J}$ projected into the plane of the sky.

For the special case of leading-order quadrupole emission ($\tilde{h}^{lm}_s = 0$ unless $l = |m| = 2$), the real part of the above expression can be rewritten as [22]

$$h_+ = \frac{2M\eta}{D}v^2 \mathrm{Re}\left[z(\alpha-\phi;\theta,\psi,\beta)e^{2i(\Phi-\gamma)}\right] \tag{5}$$

where $z(\alpha)$ is a complex variable that captures the modulation due to precession. These effects are often split into amplitude and phase modulations [14], or the waveform is decomposed in spherical harmonics on θ and ψ

[1] The three Euler angles (α,β,γ) can be visualized as follows. Start with the source frame aligned with the detector frame, so that $\mathbf{L}$ is along the z axis of the inertial frame. Rotate by γ around z, which physically contributes only to the orbital phase. Then rotate by β around y, which inclines $\mathbf{L}$ relative to z. The final rotation is by α around z, which accounts for the precession of $\mathbf{L}$ around $\mathbf{J}$. Our angle sign convention follows [24] and disagrees with [28], who use an adjoint representation.

[5]. Instead, it is possible to write

$$z(\alpha) = \sum_{m=-2}^{2} \left({}^{-2}Y_{2,m}(\beta,\alpha)\ R_m(\theta,\psi) \right) = \sum_m z_m e^{im\alpha} \tag{6}$$

where the latter expression implicitly defines the coefficients z_m. The z_m are normalized so $z_m = \delta_{m,2}$ when $\theta = \psi = 0$. In this sum, each term is proportional to $e^{im\alpha}$ and is therefore modulated at a harmonic of the precession frequency; multiplying the leading-order $\exp 2i\Phi$, each term therefore produces a sideband, offset from the carrier frequency. In general the R_m are given by

$$R_m = \sqrt{\frac{4\pi}{5}} \times \left[e^{-2i\psi}\ {}^{(-2)}Y_{2m}(\theta,0) + e^{2i\psi}\ {}^{(-2)}Y_{2-m}(\theta,0) \right] . \tag{7}$$

While this expression applies in general, we substantially increase its value by adopting coordinates with a *hierarchy of timescales*, so Φ changes on an orbital timescale; α on a radiation reaction timescale; and β and v on the inspiral timescale. This separation of timescales occurs naturally in single-spin BH-NS binaries. To an excellent approximation, a single-spin binary undergoes simple precession [14], where the total angular momentum direction is fixed; on short (precession) timescales the orbital angular momentum precesses around $\hat{J}$ at a uniform rate; and on longer (inspiral) timescales the precession cone opening angle gradually increases. Adopting coordinates aligned with J, the polar angle β is identified as the precession cone opening angle [22] and therefore changes slowly.

In the co-rotating frame, orbital evolution can be calculated using the instantaneous binary's binding energy $E(v)$ and the gravitational wave flux $\mathcal{F}(v)$ [6]:

$$\frac{dt}{dv} = -\frac{dE/dv}{\mathcal{F}} . \tag{8}$$

Both $E(v)$ and $\mathcal{F}(v)$ are known as post-Newtonian expansions in the velocity v. These are currently known to order v^7 past the leading order for non-spinning terms and v^5 in terms involving spin (note that terms at order v^6 and beyond may also have terms containing powers of $\log v$). Since dt/dv is already approximate, we are not unjustified in expanding dt/dv as a power series in v, to the same order as we know the flux and energy, to find $t(v)$ as a closed-form series in v. We use the additional relation for the orbital phase

$$\frac{d\Phi}{dv} = \frac{dt}{dv}\frac{v^3}{M} \tag{9}$$

to obtain $\Phi(v)$ in closed form. The gravitational-wave frequency is related to the velocity by $v = (\pi M f)^{1/3}$ by Kepler's Law and the fact that the gravitational-wave frequency is twice the orbital frequency.

For the special case of BH-NS binaries, we can analytically solve the spin-precession equations that determine $\vec{L}$ and hence α, β, and ζ as functions of the velocity. These three angles can be substituted into the rotation operator to transform the waveform from the precessing source frame into the inertial detector frame.

The leading order ("Newtonian") expression for the orbital angular momentum is

$$\mathbf{L} = \frac{m_1 m_2}{v}\hat{\mathbf{L}}_N , \tag{10}$$

and for the total angular momentum is

$$\mathbf{J} = \mathbf{L} + \mathbf{S}_1 . \tag{11}$$

The magnitude of the spin is conserved. In terms of the dimensionless spin, it is $\mathbf{S}_1 = m_1^2 \chi \hat{\mathbf{S}}$. The angle between $\mathbf{L}$ and $\mathbf{S}_1$ does not evolve in the single-spin case, giving the conserved quantity $\kappa = \mathbf{L}_N \cdot \mathbf{S}_1$. We can additionally define two ratios

$$\gamma \equiv \frac{|\mathbf{S}_1|}{|\mathbf{L}|} = \left(\frac{m_1\chi}{m_2}\right) v ; \tag{12}$$

$$\Gamma_J \equiv |\mathbf{J}|/|\mathbf{L}| = \sqrt{1 + 2\kappa\gamma + \gamma^2} . \tag{13}$$

The global behavior of Γ_J is not well-fit by a single low-order polynomial, as it approaches 1 at $v \to 0$ and is proportional to v in the limit of large v. As a result, expressions involving Γ_J are generally not well-fit by a standard PN expansion.

In terms of these quantities, the opening angle β of the cone swept out by $\mathbf{L}$ (the "precession cone") is

$$\cos\beta \equiv \hat{\mathbf{L}}_N \cdot \hat{\mathbf{J}} = \frac{1 + \kappa\gamma}{\Gamma_J} . \tag{14}$$

The time evolution of $\hat{\mathbf{L}}_N$ is given by

$$\frac{d\hat{\mathbf{L}}_N}{dt} = \Omega_p\ \hat{\mathbf{J}} \times \hat{\mathbf{L}}_N , \tag{15}$$

which causes $\hat{\mathbf{L}}_N$ to rotate around $\mathbf{J}$ with an angular rate of

$$\Omega_p = \eta\left(2 + \frac{3m_2}{2m_1}\right) v^5\ \Gamma_J . \tag{16}$$

The definitions of α and ζ are then

$$\dot{\alpha} = \Omega_p , \tag{17}$$

$$\dot{\zeta} = \dot{\alpha}\cos\beta = \Omega_p \cos\beta . \tag{18}$$

Substituting the definitions used above leads to a simple expression

$$\alpha(v) = \eta\left(2 + \frac{3m_2}{2m_1}\right) \int v^5\ \Gamma_J \left(\frac{dt}{dv}\right) \mathrm{d}v \tag{19}$$

$$\zeta(v) = \eta\left(2 + \frac{3m_2}{2m_1}\right) \int v^5 (1 + \kappa\gamma) \left(\frac{dt}{dv}\right) \mathrm{d}v . \tag{20}$$

We use the TaylorT2 [6] expression for dt/dv as a power series in v to express α and ζ as integrals in v rather than in t. Despite the non-polynomial behavior in Γ_J, both integrals can be evaluated term-by-term to produce closed-form expressions for $\alpha(v)$ and $\zeta(v)$ at least to 3 PN order, where terms of the form $v^n \log v$ appear. These closed-form expressions are provided as supplementary online material, and in the `lalsimulation` code. Since α, ζ and the orbital phase [Eq. (9)] influence each terms' phase in a similar way but α, ζ depend on integrals over dt/dv multiplied by at least v^2, at least two fewer terms in a post-Newtonian series expansion for dt/dv are needed to reproduce the precessional dynamics of α, ζ at the accuracy needed.

For nonprecessing signals, a commonly-used approximation to a Fourier transform is provided by the stationary-phase approximation, which for a real-valued signal $\mathrm{Re} A \exp(i\bar{\Phi}(t))$ has the form $A(v(f))/2\sqrt{id^2\Phi/dt^2} \exp i\psi(f)$, where where $\psi(f) = 2\pi f t(v) - 2\bar{\Phi}(v)$; see, e.g., Eq. (2.18) in [19]. Using this approximation, the Fourier transform of our modulated waveform [Eq. (5)] can be efficiently computed term-by-term, using the phase $\bar{\Phi}_m = 2(\Phi - \gamma) + m\alpha$. Following custom in gravitational wave data analysis, we approximate the stationary phase amplitude factor $\propto 1/\sqrt{d^2\bar{\Phi}/dt^2} \simeq (d^2\Phi/dt^2)^{-1/2}$ by its leading order term, proportional to $f^{-7/6}$. We therefore find

$$\bar{h}_+(f) \simeq \frac{2M\eta}{D\sqrt{d^2\Phi/dt^2}} v^2 \sum_m z_m e^{2i(\Psi-\gamma)+im\alpha} \tag{21}$$

$$\simeq \frac{2\pi\mathcal{M}_c^2}{D}\sqrt{\frac{5}{96\pi}}(\pi\mathcal{M}f)^{-7/6} \sum_m z_m e^{2i(\Psi-\gamma)+im\alpha} \tag{22}$$

where the expressions for $\alpha(v(f))$ and $\gamma(v(f))$ follow by explicit substitution.

Our fast, faithful frequency-domain waveform provides an efficient, easily-understood, and analytically-tractable way to represent the gravitational wave signal from generic precessing BH-NS binaries. Our method generalizes naturally to include higher harmonics. As noted above, this method allows for fast, accurate analytic Fisher matrices, to estimate the performance of parameter estimation. More broadly, overlaps between generic single-spin sources can both be calculated and understood analytically. As a concrete example, Brown et al. [22] evaluated the overlap between a generic precessing signal and a nonprecessing search template. In our representation, each sideband has a unique time-frequency trajectory, offset from the "chirp" of the orbital frequency versus time. Hence, the best fit between a nonprecessing model will occur when the nonprecessing model's time-frequency path lies on *one* of the sidebands; a nonprecessing search misses all power, except that associated with the optimal sideband; and the best-fitting match will be proportional to $|z_m|$, reproducing their general result. Finally, our model suggests several strategies for a viable search for precessing single-spin signals. First and foremost, our model allows us to identify precisely which masses, spins, and viewing orientations would benefit from a multi-modal search. Second, the functional form of z_m ensures that rarely more than two m will contribute significantly to the signal. Each of the harmonics is essentially a nonprecessing waveform and will likely match well an aligned-spin waveform with different parameters. An analytic aligned spin template bank was presented in [29]. Extending the boundaries of the bank would allow it to pick up each of the harmonics as an isolated spot of power. A promising search strategy would be to recombine the power from two (or more) spots of power and so recover most of the power in the precessing signal. Third, our expression simplifies the form of and overlaps between the l, m modes of precessing signals. Combined with a massive reduction in computational cost, our model may allow effective searches for generic precessing systems, using physical templates and maximizing over source orientation [5].

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